

Stable skyrmions from extra dimensions

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Abstract

We show that skyrmions arising from compact five dimensional models have stable sizes. We numerically obtain the skyrmion configurations and calculate their size and energy. Although their size strongly depends on the magnitude of localized kinetic-terms, their energy is quite model-independent ranging between 50 – 65 times F_π^2/m_ρ , where F_π is the Goldstone decay constant and m_ρ the lowest Kaluza-Klein mass. These skyrmion configurations interpolate between small 4D YM instantons and 4D skyrmions made of Goldstones and a massive vector boson. Contrary to the original 4D skyrmion and previous 5D extensions, these configurations have sizes larger than the inverse of the cut-off scale and therefore they are trustable within our effective 5D approach. Such solitonic particles can have interesting phenomenological consequences as they carry a conserved topological charge analogous to baryon number.

1 Introduction

Non-linear σ -models can have topological stable configurations known as skyrmions [1]. For Goldstone fields parametrizing the coset G/H , skyrmions can exist always that $\pi_3(G/H) \neq 0$. Nevertheless, the existence of non-singular configurations cannot be determined within the non-linear σ -model, since their size strongly depends on the UV-completion of the model at scales around $4\pi F_\pi$, where F_π is the Goldstone decay constant.

Five dimensional gauge theories provide UV-completions of the non-linear σ -models.¹ Any compact 5D gauge theory, in which the bulk gauge symmetry G is broken down to H at one boundary and to nothing at the other, is described at low-energies by a 4D non-linear G/H σ -model with $F_\pi \sim \sqrt{M_5/L}$, where M_5 is the inverse squared of the 5D gauge coupling and L is the compactification scale in conformal coordinates. The cut-off scale of the 5D gauge theories is estimated to be $\Lambda_5 \sim 24\pi^3 M_5$. Therefore, provided that $M_5 \gtrsim 1/L$, these theories are valid up to energies much larger than the scale $4\pi F_\pi$. Being this the case, compact five dimensional gauge theories allows us to address the question of the stability of the skyrmion configurations and calculate their properties. This is the subject of this paper.

We will calculate numerically the skyrmion configuration arising from compact extra dimensional models, concentrating in flat and AdS spaces. We will show that these configurations have stable sizes, calculable within the regime of validity of our effective 5D theory. The size, however, depends not only on M_5 and L , but also on the coefficient of the lowest higher-dimensional operator of the 5D theory. The energy of the skyrmion will also be calculated and shown to be predicted in a narrow range.

Attempts to address the stability of the skyrmion in extra dimensional models can be found recently in the literature [2–6]. Nevertheless, we consider that none of them address fully consistently the stability of the skyrmion configuration (to determine the size within the higher-dimensional effective theory approach). Our analysis is also the first to exactly calculate the skyrmion configuration in 5D models that requires the use of numerical methods.

The interest in the existence of these topological objects is not only theoretical but also phenomenological. In the recent years there has been a lot of activity in using extra dimensions in order to break the electroweak symmetry. In these examples the existence of topologically stable particles can lead to important cosmological implications. Another

¹Strictly speaking, 5D theories provide only UV-extensions of the non-linear σ -models since they are not well-defined theories at arbitrarily high energies.

interest in skyrmion configuration arises in the context of warped 5D spaces. The AdS/CFT correspondence tells us that skyrmions in 5D are the dual of the "baryons" of 4D strongly coupled theories. Therefore studying these 5D solitons can be useful to learn about the properties of the baryons.

2 Skyrmions in compact and warped 5D spaces

We will be looking for skyrmion configurations arising from the global symmetry breaking pattern $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$. These configurations will still be valid for any breaking $G \rightarrow H$ always that G and H contains respectively $SU(2)_L \times SU(2)_R$ and $SU(2)_V$ as a subgroup. This includes, for example, the case $SU(N)_L \times SU(N)_R \rightarrow SU(N)_V$.

The class of 5D theories that we will be considering corresponds to the following one. They are $SU(2)_L \times SU(2)_R$ gauge theories with metric $ds^2 = a(z)^2 (dx_\mu dx^\mu - dz^2)$, where x^μ represent the usual 4 coordinates (4D indeces are raised and lowered with the flat Minkowski metric) and z , which runs in the interval $[z_{UV}, z_{IR}]$, denotes the extra dimension. We will take $a(z) \geq a(z_{IR}) = 1$. We will denote respectively L_M and R_M , where $M = (\mu, 5)$, the $SU(2)_L$ and $SU(2)_R$ gauge connections parametrized by $L_M = L_M^a \sigma_a / 2$ and $R_M = R_M^a \sigma_a / 2$ where σ_a are the Pauli matrices. The chiral symmetry breaking is imposed on the boundary at $z = z_{IR}$ (IR-boundary) by requiring the following boundary conditions:

$$(L_\mu - R_\mu)|_{z=z_{IR}} = 0, \quad (L_{\mu 5} + R_{\mu 5})|_{z=z_{IR}} = 0, \quad (1)$$

where the 5D field strenght is defined as $L_{MN} = \partial_M L_N - \partial_N L_M - i[L_M, L_N]$ and analogously for R . At the other boundary, the UV-boundary, we impose Dirichlet conditions to all the fields:

$$L_\mu|_{z=z_{UV}} = R_\mu|_{z=z_{UV}} = 0. \quad (2)$$

The Kaluza–Klein (KK) and holographic description of the 5D theory described above has been extensively studied in the literature [7, 8]. From a 4D point of view, the theory resembles to large- N_c QCD where the Goldstone bosons or pions are associated to the fifth gauge field component, and the ρ and a_1 resonances correspond to the gauge KK-states. At low-energies all these 5D theories are described by an $(SU(2)_L \times SU(2)_R)/SU(2)_V$ non-linear σ -model.

We are looking for static finite energy solutions of the classical equations of motion (EOM) of the above 5D theory. 4D Lorentz invariance allows us to take the ansatz $L_0 = R_0 = 0$, $L_{\hat{\mu}} = L_{\hat{\mu}}(\mathbf{x}, z)$ and $R_{\hat{\mu}} = R_{\hat{\mu}}(\mathbf{x}, z)$, where 0 and $\hat{\mu}$ label, respectively, the temporal and spacial coordinates while \mathbf{x} denotes ordinary 3-space. Starting from the usual 5D YM action the

energy of this field configuration reads

$$E = \int d^3x \int_{z_{\text{UV}}}^{z_{\text{IR}}} dz a(z) \frac{M_5}{2} \text{Tr} [L_{\hat{\mu}\hat{\nu}} L^{\hat{\mu}\hat{\nu}} + R_{\hat{\mu}\hat{\nu}} R^{\hat{\mu}\hat{\nu}}] , \quad (3)$$

where the indices are now raised and lowered by the Euclidean 4D metric. Finding solutions to the EOM is the same as minimizing the energy functional Eq. (3), which closely resembles the Euclidean YM action in 4D. Our problem is therefore very similar to the one of finding $SU(2)$ instantons, even though, as we will see later, there are some important differences which will not allow us to find an analytic solution.

The topological charge of the soliton will be defined by

$$Q = \frac{1}{32\pi^2} \int d^3x \int_{z_{\text{UV}}}^{z_{\text{IR}}} dz \epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \text{Tr} [L^{\hat{\mu}\hat{\nu}} L^{\hat{\rho}\hat{\sigma}} - R^{\hat{\mu}\hat{\nu}} R^{\hat{\rho}\hat{\sigma}}] , \quad (4)$$

which is the difference between the L and R instanton charges. In order to show that Q is a topological integer number, and with the aim of making the relation with the skyrmion more precise, it is convenient to go to the axial gauge $L_5 = R_5 = 0$. The latter can be easily reached, starting from a generic gauge field configuration, by means of a Wilson-line transformation. In the axial gauge both boundary conditions Eqs. (1) and (2) cannot be simultaneously satisfied. Let us then keep Eq. (1) but modify the UV-boundary condition to

$$\tilde{L}_i|_{z=z_{\text{UV}}} = i U(\mathbf{x}) \partial_i U(\mathbf{x})^\dagger, \quad \tilde{R}_i|_{z=z_{\text{UV}}} = 0, \quad (5)$$

where \tilde{L}_i and \tilde{R}_i are the gauge fields in the axial gauge and i runs over the 3 ordinary space coordinates. The field $U(\mathbf{x})$ in the equation above precisely corresponds to the Goldstone field in the 4D interpretation [9] once a static Ansatz is taken. By using a form notation [10] $A = -i A_{\hat{\mu}} dx^{\hat{\mu}}$ and remembering that $F \wedge F = d\omega_3(A)$, the 4D integral in Eq. (4) can be rewritten as an integral of the third Chern–Simons form $\omega_3(A)$ on the 3D boundary of the space:

$$Q = \frac{1}{8\pi^2} \int_{3D} [\omega_3(\tilde{L}) - \omega_3(\tilde{R})] . \quad (6)$$

The contribution to Q coming from the IR-boundary vanishes as the L and R terms in Eq. (6) cancel each other due to Eq. (1). This is crucial for Q to be quantized and it is the reason why we have to choose the relative minus sign among the L and R instanton charges in the definition of Q . At the $\mathbf{x}^2 \rightarrow \infty$ boundary, the contribution to Q also vanishes since in the axial gauge $\partial_5 A_i = 0$ (in order to have $F_{5i} = 0$). We are then left with the UV-boundary which we can topologically regard as the 3-sphere S_3 . Therefore, we find

$$Q = -\frac{1}{8\pi^2} \int_{\text{UV}} \omega_3 [\tilde{L}_i (= i U \partial_i U^\dagger)] = \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{Tr} [U \partial_i U^\dagger U \partial_j U^\dagger U \partial_k U^\dagger] \in \mathbb{Z}. \quad (7)$$

The charge Q is equal to the Cartan–Maurer integral invariant for $SU(2)$ which is an integer. Then solutions with nonzero Q , if they exist, cannot trivially correspond to a pure gauge configuration and they must have positive energy. Moreover, the particles associated to solitons with $Q = \pm 1$ will be stable given that they have minimal charge. Eq. (7) also tells us that the non-trivial configuration $U(\mathbf{x})$ corresponds to a 4D skyrmion with B being the baryon number. In a general gauge, the value of $U(\mathbf{x})$ will be given by

$$U(\mathbf{x}) = P \left\{ \exp \left[-i \int_{z_{UV}}^{z_{IR}} dz' R_5(\mathbf{x}, z') \right] \right\} \cdot P \left\{ \exp \left[i \int_{z_{UV}}^{z_{IR}} dz' L_5(\mathbf{x}, z') \right] \right\}, \quad (8)$$

where P indicates path ordering. From a KK perspective, the 5D soliton that we are looking for can be considered to be a 4D skyrmion made of Goldstone bosons and the massive tower of KK gauge bosons.

We want to find a numerical solution to the 4D EOM arising from the energy functional of Eq.(3) with suitable boundary conditions enforcing $Q = 1$. To do that, as we will now discuss, the axial gauge is not an appropriate choice. We will then restart with in a gauge-independent way and specify later on the new gauge-fixing condition. It is expected on general grounds that the solution will be maximally symmetric, *i.e.* invariant under all symmetry transformations which are compatible with the boundary conditions. We impose, first of all, invariance under "cylindrical" transformations, *i.e.* combined $SU(2)$ gauge and 3D spacial rotations. This corresponds to the following Ansatz [11]:

$$\begin{aligned} L_j^a &= -\frac{1 + \phi_2^L(r, z)}{r^2} \epsilon_{jak} x_k + \frac{\phi_1^L(r, z)}{r^3} (r^2 \delta_{ja} - x_j x_a) + \frac{A_1^L(r, z)}{r^2} x_j x_a, \\ L_5^a &= \frac{A_2^L(r, z)}{r} x^a, \end{aligned} \quad (9)$$

and similarly for $R_{\hat{\mu}}$. We have reduced our original 4D problem to a much simpler 2 dimensional one, being $x^{\bar{\mu}} = \{r, z\}$ (where $r = \sqrt{\mathbf{x}^2}$) the 2D coordinates. The Ansatz has partially fixed the gauge leaving only a $U(1)_L \times U(1)_R$ subgroup. The allowed transformations are those which preserve the cylindrical symmetry and have the form $g_{L,R} = \exp[i \alpha_{L,R}(r, z) x^a \sigma_a / (2r)]$. Under this symmetry $A_{\bar{\mu}}^{L,R}$ transform as gauge fields ($A_{\bar{\mu}}^{L,R} \rightarrow A_{\bar{\mu}}^{L,R} + \partial_{\bar{\mu}} \alpha_{L,R}$) and $\phi_{L,R} = \phi_1^{L,R} + i \phi_2^{L,R}$ are complex scalars of charge $+1$.

In addition to the global symmetries, our problem has also discrete ones. Those are the $L \leftrightarrow R$ interchange and ordinary parity $\mathbf{x} \rightarrow -\mathbf{x}$. The charge Q is odd under each of these transformations, so we cannot impose our solution to be separately invariant under both, but only under their combined action. We can then further reduce our Ansatz by imposing

$$L_{\hat{\mu}}(\mathbf{x}, z) = R_{\hat{\mu}}(-\mathbf{x}, z):$$

$$\begin{aligned} A_1 &\equiv A_1^R = -A_1^L, & A_2 &\equiv A_2^R = -A_2^L, \\ \phi_1 &\equiv \phi_1^R = -\phi_1^L, & \phi_2 &\equiv \phi_2^R = \phi_2^L. \end{aligned} \quad (10)$$

Our solution is now fully specified by 4 real 2D functions $A_{\bar{\mu}}$ and ϕ . We have a residual $U(1)$ invariance corresponding to $g_L^\dagger = g_R = \exp[i \alpha(r, z) x^a \sigma_a / (2r)]$ under which $A_{\bar{\mu}}$ is the gauge field and ϕ has charge +1.

By substituting the Ansatz Eqs. (9) and (10) into the energy Eq. (3), one finds

$$E = 16\pi \int_0^\infty dr \int_{z_{UV}}^{z_{IR}} dz M_5 a(z) \left[\frac{1}{2} |D_{\bar{\mu}} \phi|^2 + \frac{1}{8} r^2 F_{\bar{\mu}\bar{\nu}}^2 + \frac{1}{4r^2} (1 - |\phi|^2)^2 \right]. \quad (11)$$

In flat space, $a(z) = 1$, this corresponds to a 2D Abelian Higgs model (with metric $g^{\bar{\mu}\bar{\nu}} = r^2 \delta^{\bar{\mu}\bar{\nu}}$). For a general warp factor we have, however, a non-metrical theory. Substituting the Ansatz in the topological charge Eq. (4) we find

$$Q = \frac{1}{2\pi} \int_0^\infty dr \int_{z_{UV}}^{z_{IR}} dz \epsilon^{\bar{\mu}\bar{\nu}} \left[\partial_{\bar{\mu}} (-i\phi^* D_{\bar{\nu}} \phi + h.c.) + F_{\bar{\mu}\bar{\nu}} \right]. \quad (12)$$

The charge can be written, as it should, as an integral over the $1D$ boundary of the $2D$ space. We will choose the boundary conditions in such a way that the first term of Eq. (12) vanishes, and therefore Q will coincide with the magnetic flux, *i.e.* the topological charge of the Abelian Higgs model.

In order to solve numerically the EOM associated with Eq. (11), they must be recast in the form of a 2D system of non-linear Elliptic Partial Differential Equations (EPDE). The numerical resolution of the 2D EPDE boundary value problem has indeed been widely studied and very simple and powerful packages exist. This is the reason why we cannot work in the axial gauge, since the EOM for A_1 is not elliptic in this gauge. We will impose the 2D Lorentz gauge condition $\partial_{\bar{\mu}} A^{\bar{\mu}} = 0$. The equations for $A_{\bar{\nu}}$ become $J^{\bar{\nu}} = \partial_{\bar{\mu}} (r^2 a F^{\bar{\mu}\bar{\nu}}) = r^2 a \square A^{\bar{\nu}} + \partial_{\bar{\mu}} (r^2 a) F^{\bar{\mu}\bar{\nu}}$, where \square is the 2D Laplacian and J is the current of the field ϕ . In this way we have a system of 4 EPDE and 4 real unknown functions which we can determine numerically. Since we are counting the two gauge field components as independent functions, one can wonder whether the gauge-fixing condition is satisfied. Notice however that by taking the $\partial_{\bar{\nu}}$ derivative of the EOM for $A_{\bar{\nu}}$ and observing that the current is conserved, $\partial_{\bar{\nu}} J^{\bar{\nu}} = 0$, on the solutions of the EOM for ϕ , we find an EPDE for $\partial_{\bar{\nu}} A^{\bar{\nu}}$ which reads $[(r^2 a) \square + \partial_{\bar{\mu}} (r^2 a) \partial^{\bar{\mu}}] (\partial_{\bar{\nu}} A^{\bar{\nu}}) = 0$. This equation has a unique solution once the boundary conditions are given. If we impose $\partial_{\bar{\nu}} A^{\bar{\nu}} = 0$ at the boundaries, the gauge will then be automatically maintained everywhere in the bulk.

We will solve the EOM in the rectangle ($z \in [z_{\text{UV}}, z_{\text{IR}}]$, $r \in [0, R]$) and we will numerically take the limit $R \rightarrow \infty$. At the three sides $z = z_{\text{IR}}$, $z = z_{\text{UV}}$ and $r = R$ we impose the following boundary conditions

$$z = z_{\text{IR}} : \begin{cases} \phi_1 = 0 \\ \partial_2 \phi_2 = 0 \\ A_1 = 0 \\ \partial_2 A_2 = 0 \end{cases}, \quad z = z_{\text{UV}} : \begin{cases} \phi_1 = 0 \\ \phi_2 = -1 \\ A_1 = 0 \\ \partial_2 A_2 = 0 \end{cases}, \quad r = R : \begin{cases} \phi = -i e^{i\pi z/L} \\ \partial_1 A_1 = 0 \\ A_2 = \frac{\pi}{L} \end{cases}, \quad (13)$$

where L is the conformal length of the extra dimension:

$$L = \int_{z_{\text{UV}}}^{z_{\text{IR}}} dz = z_{\text{IR}} - z_{\text{UV}}. \quad (14)$$

Few comments are in order. The conditions on the IR and UV boundary for $\phi_{1,2}$ and A_1 come respectively from Eqs. (1) and (2), while the condition for A_2 arises from the gauge fixing. On the boundary at $r = R$ we have chosen the fields ϕ and A_2 with a nontrivial profile consistent, however, with $E \rightarrow 0$ at $R \rightarrow \infty$; the condition for A_1 comes again from the gauge fixing. Our choice of boundary conditions is such that the charge Eq. (12) receives contributions only from the boundary at $r = R$. The $r = 0$ boundary of our rectangular domain is special, given that the EOM become singular there. The software which we will employ (FEMLAB 3.1) permits us to extend the domain up to $r = 0$ because it never uses the EOM on the boundary lines; the equations on that points are provided by the boundary conditions. Some special care is required, however, to enforce the program to converge to a regular solution. We find that to impose regularity the following conditions are needed:

$$r = 0 : \begin{cases} \phi_1/r \rightarrow A_1 \\ (1 + \phi_2)/r \rightarrow 0 \\ A_2 \rightarrow 0 \\ \partial_1 A_1 = 0 \end{cases}. \quad (15)$$

The conditions for $\phi_{1,2}$ and A_2 are extracted from Eq. (9) by requiring the gauge fields to be well defined 5D vectors, while the condition for A_1 is the gauge fixing. To impose these conditions and obtain regular solutions we define the rescaled fields $\chi_{1,2}$: $\phi_1 = r\chi_1$ and $\phi_2 = -1 + r\chi_2$. The boundary condition Eq. (15) reads now $\chi_1 = A_1$, $\chi_2 = 0$ and $A_2 = 0$, plus the gauge-fixing condition $\partial_1 A_1 = 0$. The rescaled fields $\chi_{1,2}$, together with $A_{1,2}$, are actually the ones used in our numerical computations.

Any gauge-invariant information about the 5D soliton such as the energy and charge densities Eqs. (11) and (12) respectively can be directly extracted with our procedure. It would be also interesting, however, to know also what the profile of the skyrmion is. To this end we have to go to the axial gauge, as we discussed above. From Eq. (8) we have

$$U(\mathbf{x}) = \exp \left[-i \sigma_i x^i / r \int_{z_{\text{UV}}}^{z_{\text{IR}}} dz' A_2(r, z') \right] \equiv \exp [i f(r) \sigma_i x^i / r]. \quad (16)$$

We immediatly see that our boundary conditions imply $f(0) = 0$ and $f(\infty) = -\pi$, as it should be for a skyrmion in which $f(r)$ undergoes a $-\pi$ variation when r goes from 0 to ∞ .

When applying the above described numerical method, however, we do not find any solution for either a flat or warped extra dimension. The 5D theory which we are considering does not possess any non-singular soliton and the reason is the following [3, 4]. The energy Eq. (3) of a 5D soliton configuration is bounded from below:

$$E \geq \int d^3x \int_{z_{UV}}^{z_{IR}} dz a(z) \frac{M_5}{4} \text{Tr} |\epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} F^{\hat{\mu}\hat{\nu}} F^{\hat{\rho}\hat{\sigma}}| \geq 8\pi^2 M_5 |Q| , \quad (17)$$

where we take taken, for simplicity, the Ansatz $L_{\hat{\mu}\hat{\nu}}(\mathbf{x}, z) = R_{\hat{\mu}\hat{\nu}}(-\mathbf{x}, z) \equiv F_{\hat{\mu}\hat{\nu}}(\mathbf{x}, z)$ and in the last inequality we have used $a(z) \geq a(z_{IR}) = 1$. The first inequality can be saturated if $F_{\hat{\mu}\hat{\nu}}$ is self-dual or antiself-dual, while the the lower bound $8\pi^2 M_5 |Q|$ is approached (in warped spaces) when these (anti)self-dual configurations are centered at $z = z_{IR}$ and tend to zero size (since $a(z)$ has its minimum at $z = z_{IR}$). This is the case of a 4D YM instanton configuration of infinitesimal size, $\rho \rightarrow 0$, with the identification of the Euclidean time t_E with z , and centered at $z = z_{IR}$. This is clear since the limit $\rho \rightarrow 0$ is equivalent to keeping the instanton size fixed and taking to zero all other dimensional parameters, the curvature and the extra dimensional size , $a(z) \rightarrow 1$ and $z_{UV} \rightarrow -\infty$ respectively. In this limit the energy Eq. (3) corresponds to the 4D Euclidian action that is minimized by an instanton of arbitrary size. This means that the 5D soliton that we are looking for is a singular instanton configuration.

An alternative way to see this is by explicitly calculating the energy of the instanton configuration in the limit of small size. This is done in the Appendix. For a flat space we obtain

$$E(\rho) \simeq 8\pi^2 M_5 \left[1 + \frac{1}{4} \left(\frac{\rho}{L} \right)^4 \right] , \quad (18)$$

while for the AdS space we have

$$E(\rho) \simeq 8\pi^2 M_5 \left[1 + \frac{\rho}{2L} \right] . \quad (19)$$

In both cases we see that the minimum of the energy is achieved for $\rho \rightarrow 0$.

2.1 IR-boundary terms

The problem of stabilizing the radius of the soliton configuration can be overcome if there is a repulsive force on the IR-boundary that pushes the instanton to large sizes. The simplest possibility is to introduce an IR-boundary kinetic term for the gauge fields:

$$S_{IR} = - \int d^4x \frac{\alpha}{2} \text{Tr} [L_{\mu\nu} L^{\mu\nu} + R_{\mu\nu} R^{\mu\nu}] \Big|_{z=z_{IR}} . \quad (20)$$

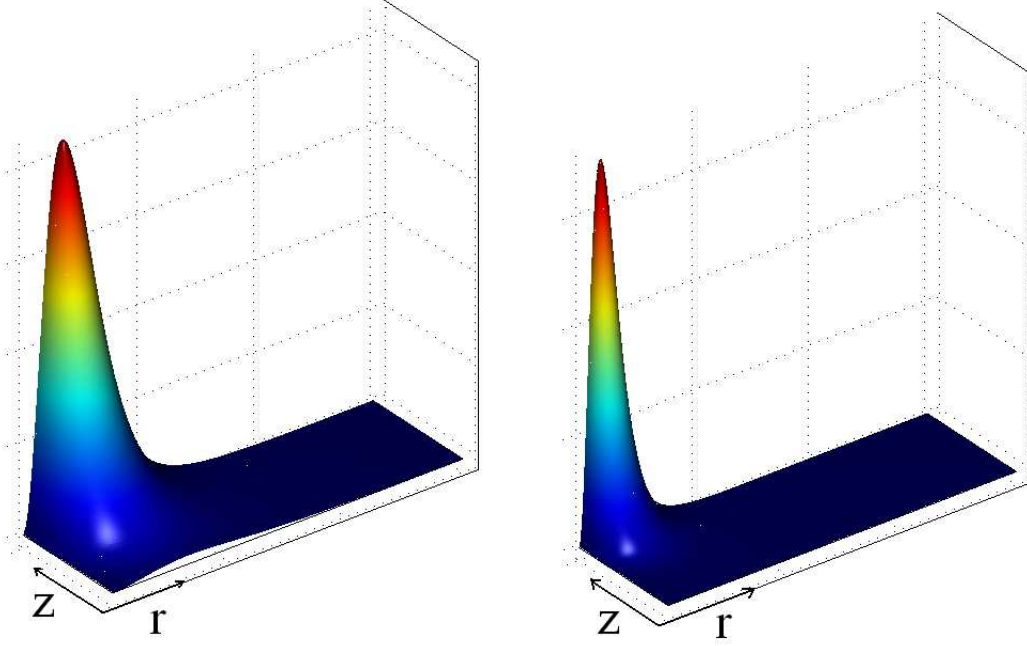


Figure 1: We plot the bulk energy density Eq. (11) of our numerical solution for flat (left panel) and AdS (right panel) space. Both plots are obtained for $\alpha/(LM_5) = 0.1$.

The presence of this term is expected in these theories since it is unavoidably induced at the loop level. An NDA estimate gives $\alpha \sim 1/(16\pi^2)$. Furthermore, as we will show in section 2.3, Eq. (20) is the lowest higher-dimensional operator of our effective 5D theory and therefore the most important one beyond Eq. (3).

Using Eqs. (9) and (10) and the fact that $\phi_1 = A_1 = 0$ at $z = z_{\text{IR}}$, we can write the boundary energy Eq. (20) as

$$E_{\text{IR}} = 8\pi\alpha \int_0^\infty dr \left[(\partial_1 \phi_2)^2 + \frac{1}{2r^2} (1 - \phi_2^2)^2 \right] \Big|_{z=z_{\text{IR}}} . \quad (21)$$

We can see that this term favors large instanton configurations by substituting the instanton Eq. (36) in Eq. (21). We obtain

$$E_{\text{IR}}(\rho) = 6\pi^2 \frac{\alpha}{\rho} , \quad (22)$$

that grows for small ρ . Therefore we expect that the presence of Eq. (21) will guarantee the existence of non-singular 5D solitons. These configurations cannot be found analytically and for this reason we have to rely on numerical analysis. The effect of Eq. (21) corresponds to a change of the IR-boundary condition for ϕ_2 . Instead of that in Eq. (13), we have now

$$\partial_2 \phi_2 \Big|_{z=z_{\text{IR}}} = \frac{\alpha}{M_5} \left[\partial_1^2 \phi_2 + \frac{1}{r^2} \phi_2 (1 - \phi_2^2) \right] \Big|_{z=z_{\text{IR}}} . \quad (23)$$

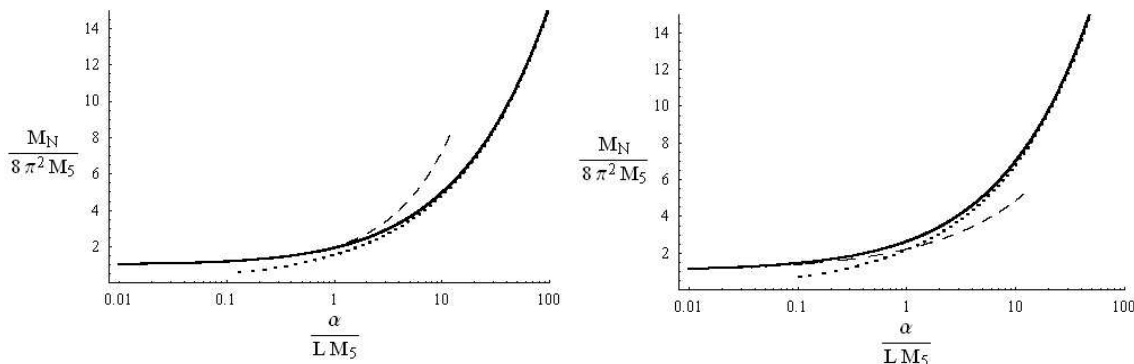


Figure 2: Left panel: the thick line represents the soliton energy which we obtained numerically in the case of flat space. This is well approximated by Eq. (30) (dashed line) and Eq. (33) (dotted line) for, respectively, small and large values of $\alpha/(LM_5)$. Right panel: the same plot for AdS; the approximated curves are provided by Eq. (31) (dashed line) and Eq. (33) (dotted line).

2.2 Numerical results

To find numerically the 5D soliton configuration we use the software package FEMLAB 3.1 [12]. This is a numerical package for solving EPDE based on the finite element method. We have concentrated on two different spaces:

$$\text{Flat:} \quad a(z) = 1, \quad (24)$$

$$\text{AdS:} \quad a(z) = \frac{z_{\text{IR}}}{z} \quad \text{with} \quad z_{\text{UV}} \rightarrow 0. \quad (25)$$

Fig. 1 shows an example of the numerical results that we get; for $\alpha/(LM_5) = 0.1$ we plot the bulk energy density of the soliton in flat and AdS space. Notice that the size of the AdS soliton is sensibly smaller than the flat one and that the energy density vanishes at $z = z_{\text{UV}}$ in the case of AdS while it does not in flat space. This is consistent with the holographic CFT interpretation of the AdS soliton. Being localized at the IR-boundary, the soliton is a composite state of the CFT.

In Fig. 2 we present the total energy of the soliton configuration $E_{\text{total}} = E + E_{\text{IR}}$ for different values of $\alpha/(M_5L)$. This approximately corresponds to the mass of the soliton ² $M_N \simeq E_{\text{total}}$. Although the energy of the soliton is quite sensitive to $\alpha/(M_5L)$ for large values of this quantity, this is not the case when E_{total} is expressed as a function of the PGB

²Here we work at the semi-classical level neglecting corrections due to the quantization of the soliton.

decay constant F_π and the mass of the lowest KK-state m_ρ that are calculated to be

$$F_\pi^2 = \frac{2M_5}{L}, \quad m_\rho \simeq \frac{\pi}{2L} \left(1 + \frac{\pi^2 \alpha}{4M_5 L} \right)^{-1/2} \quad \text{for flat space,} \quad (26)$$

$$F_\pi^2 = \frac{4M_5}{L}, \quad m_\rho \simeq \frac{3\pi}{4L} \left(1 + \frac{9\pi^2 \alpha}{32M_5 L} \right)^{-1/2} \quad \text{for AdS space,} \quad (27)$$

where F_π is normalized such that the coefficient in front of the kinetic term of the Goldstone is $F_\pi^2/4$. In Fig. 3 we show the ratio $M_N m_\rho / F_\pi^2$ for different values of α . We see that this ratio only varies a 20% and stays in the range 50–65, resulting in a relatively model-independent prediction of the soliton mass. We define the radius ρ_0 of the configuration as the mean radius of the topological charge density, *i.e.*

$$\rho_0^2 = \frac{1}{2\pi} \int_0^\infty dr \int_{z_{UV}}^{z_{IR}} dz \frac{r^2 + (z - z_{IR})^2}{2} \epsilon^{\bar{\mu}\bar{\nu}} \left[\partial_{\bar{\mu}} (-i\phi^* D_{\bar{\nu}} \phi + h.c.) + F_{\bar{\mu}\bar{\nu}} \right]. \quad (28)$$

For the instanton configuration in a flat uncompactified space ρ_0 coincides with ρ . The radius ρ_0 will go to zero as $\alpha \rightarrow 0$ and will diverge for $\alpha \rightarrow \infty$. In order to show the behaviour of ρ_0 in these two limits, we plot in Fig. 4 the following combinations: $\rho_0 [M_5/(\alpha L)]^{1/2}$ for both flat and AdS space and $\rho_0 [M_5/(\alpha L^4)]^{1/5}$ for flat space only. We will show below how these behaviours can be analytically deduced.

The numerical results presented above have a simple interpretation in the limit in which the boundary term Eq. (20) dominates or not over the bulk Eq. (3). This corresponds to the limits $\alpha \gg M_5 L$ and $\alpha \ll M_5 L$ respectively. We will discuss each in turn.

$\alpha \ll M_5 L$: In this limit the energy of the soliton is dominated by the bulk contribution, and therefore we expect that the instanton configuration is a good approximation of the true solution. In this case we can find the size of the configuration by minimizing the instanton energy with respect to ρ . For flat space, the minimum of Eqs. (18) and (22) corresponds to

$$\rho_0 \simeq \left(\frac{3\alpha}{4M_5 L} \right)^{1/5} L, \quad (29)$$

that leads to the total energy

$$E_{total} \simeq 8\pi^2 M_5 \left(1 + \frac{5}{4} \left(\frac{3\alpha}{4M_5 L} \right)^{4/5} \right). \quad (30)$$

For AdS space, the minimization of Eqs. (19) and (22) gives

$$\rho_0 \simeq \sqrt{\frac{3\alpha}{2M_5 L}} L, \quad E_{total} \simeq 8\pi^2 M_5 \left(1 + \sqrt{\frac{3\alpha}{2M_5 L}} \right). \quad (31)$$

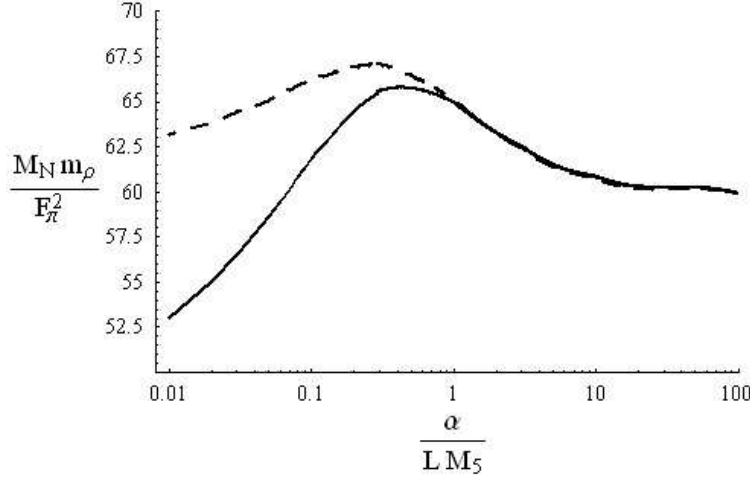


Figure 3: The ratio $M_N m_\rho / F_\pi^2$ is plotted as a function of $\alpha / (LM_5)$ for a flat (dashed line) and AdS space. The behaviour for large $\alpha / (LM_5)$ is consistent with Eq. (33) while at small $\alpha / (LM_5)$ we obtain Eq. (32).

These analytical results agree in the small α limit with the numerical ones. From Fig. 4 we see in fact that ρ_0 scales as $(\alpha / (M_5 L))^{1/5}$ and $(\alpha / (M_5 L))^{1/2}$ for flat and AdS space respectively in the limit of small α . The energy in this limit is approximately given by the instanton energy $8\pi^2 M_5$ independently of the metric or the compactification of the extra dimension as can be seen from Fig. 2. Finally, the energy can be rewritten as

$$M_N \simeq 2\pi^3 \frac{F_\pi^2}{m_\rho} \quad (\text{flat space}), \quad M_N \simeq \frac{3\pi^3}{2} \frac{F_\pi^2}{m_\rho} \quad (\text{AdS space}), \quad (32)$$

in accordance with Fig. 3.

$\alpha \gg M_5 L$: In this limit the boundary term Eq. (20) is dominant, and one finds that the effective low-energy theory below $1/L$ corresponds to the following one. It is a 4D theory with a non-linearly realized $SU(2)_L \times SU(2)_R$ global symmetry in which the $SU(2)_V$ subgroup is weakly gauged. The gauge coupling is given by $g = 1/\sqrt{\alpha}$ and the mass of the gauge boson is then $m_\rho = g F_\pi$ where F_π is the Goldstone decay constant defined above. This is valid independently of the geometry of the extra dimension. One can find this result by ordinary KK reduction that below $1/L$ leads to massless scalars $(L_5 - R_5)$, the Goldstones, plus a KK-state of $(L_\mu + R_\mu)$ with mass $\sim \sqrt{M_5 / (L\alpha)}$ —see Eqs. (26) and (27). A better understanding of the large α limit, however, can be obtained by following a “holographic” prescription and treating the bulk gauge fields and its value at the IR-boundary as distinct variables. The bulk plus the UV-boundary sector has a global $SU(2)_L \times SU(2)_R$ symmetry broken down to nothing due to the boundary condition Eq. (2). Its 4D spectrum contains the heavy states of mass of order $1/L$ and the Goldstone bosons. On the other hand, the IR-boundary (see Eq. (1) and Eq. (20)) corresponds to a gauging of only the $SU(2)_V$ subgroup of the global

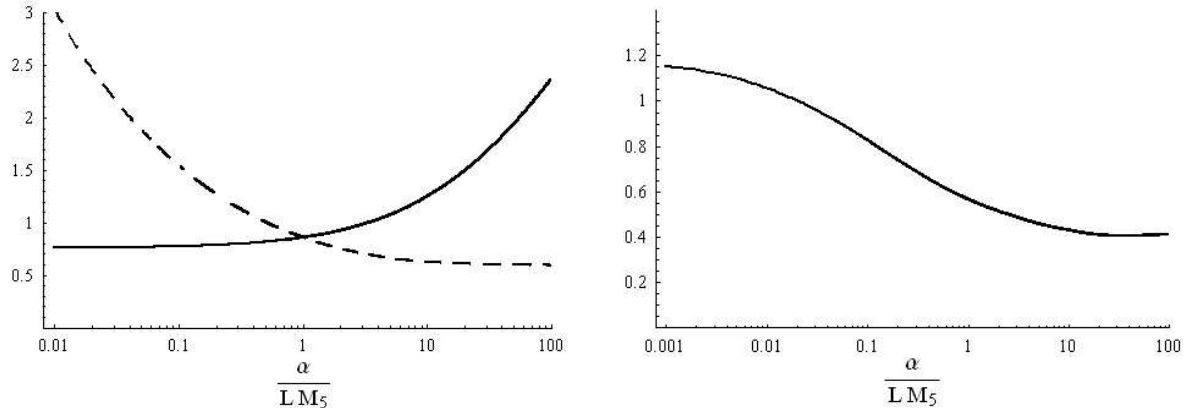


Figure 4: Left panel: $\rho_0[M_5/(\alpha L)]^{1/2}$ (dashed) and $\rho_0[M_5/(\alpha L^4)]^{1/5}$ for the case of flat space. Right panel: $\rho_0[M_5/(\alpha L)]^{1/2}$ in AdS. The limiting behaviour of those quantities is consistent with Eqs. (29), (31) and (34).

$SU(2)_L \times SU(2)_R$ with $1/g^2 = \alpha$. Therefore the spectrum of the full 5D theory below $1/L$ consists of the Goldstone bosons parametrizing the coset $(SU(2)_L \times SU(2)_R)/SU(2)_V$ with decay constant F_π plus the $SU(2)_V$ gauge bosons that acquire a mass $gF_\pi \ll 1/L$. Theories like these were considered in the past and it was shown that they have stable non-singular skyrmions [13]. The mass of the $Q = 1$ skyrmion was found to be ³

$$M_N \simeq 59.4 \frac{F_\pi^2}{m_\rho}. \quad (33)$$

In Fig. 3 one can see that our numerical value of M_N tends to Eq. (33) for large values of α . We also see from Fig. 4 that the radius of our 5D soliton scales in the large α limit as

$$\rho_0 \sim \sqrt{\alpha L/M_5} \sim 1/m_\rho, \quad (34)$$

as expected for a 4D skyrmion. Thus, we can conclude that for $\alpha \gg M_5 L$ the 5D soliton corresponds to a 4D skyrmion made of Goldstones and a massive gauge field.

Summarizing, by ranging α from 0 to ∞ , our 5D soliton configuration goes from being a small 4D instanton configuration to the 4D skyrmion of Ref. [13]. By increasing α we are effectively decreasing the contribution to the soliton of the heaviest gauge KK-states that reduces to a single KK in the limit of very large α .

2.3 Calculability in the 5D effective theory

We have found that once we include the IR term Eq. (20) into the action a regular static solution to the 5D EOM with $Q = 1$ exists. In the spirit of effective field theories, however,

³We are taking the result from Ref. [13] where for $a = m_\rho^2/(g^2 F_\pi^2) = 1$, $m_\rho \simeq 770$ GeV and $F_\pi \simeq 93$ GeV they obtain $M_N \simeq 667$ GeV.

the action also contains an infinite number of 5D higher-dimensional operators which are suppressed by inverse powers of the cut-off scale Λ_5 , and it is of crucial importance to establish how much our results are sensitive to such operators.

The contribution to the soliton energy of the higher-dimensional operators is easily estimated by substituting the soliton configuration into the operators. Since these operators have at least one more dimension of energy, their coefficients are suppressed by one more power of Λ_5^{-1} . On dimensional grounds, therefore, the contribution to the energy of the new operators carries on more powers of $1/(\Lambda_5\rho_0)$ where the radius ρ_0 is the typical size of the soliton. We must check that $1/(\Lambda_5\rho_0) \ll 1$ in such a way that a sensible perturbative expansion, analogous to the E/Λ_5 expansion for a scattering amplitude at energies $E < \Lambda_5$, can be performed. If this is the case the 5D soliton can be studied with our 5D effective theory in the sense that we can compute its properties, to a certain degree of accuracy, by only including a finite number of operators into the action.

By looking at Eqs. (29) and (31), we find that $\rho_0\Lambda_5 \sim (\Lambda_5 L)^{4/5}$ and $\rho_0\Lambda_5 \sim (\Lambda_5 L)^{1/2}$ respectively for a flat and AdS space. Therefore, as long as $\Lambda_5 L \ll 1$ (as it should for our extra-dimensional theory to make sense), we have $1/(\Lambda_5\rho_0) \ll 1$ as needed in order to trust our soliton within the 5D effective approach. We can see this in more detail by considering a specific 6-dimensional bulk operator such as $F_{MN}D_R D^R F^{MN}$ that, by the NDA counting, is suppressed by M_5/Λ_5^2 where $\Lambda_5 \sim 24\pi^3 M_5$. The contribution of this operator to the soliton energy is of order $\delta E \sim 8\pi^2 M_5/(\Lambda_5\rho_0)^2$ that should be compared with the boundary energy Eq. (22). The relative correction $\delta E/E_{\text{IR}}(\rho_0) \sim (24\pi^3\alpha\Lambda_5\rho_0)^{-1}$ plays the role of the expansion parameter. Using Eqs. (29) and (31) we get, respectively

$$(24\pi^3\alpha)^{-6/5}(\Lambda_5 L)^{-4/5} \quad \text{and} \quad (24\pi^3\alpha)^{-3/2}(\Lambda_5 L)^{-1/2}. \quad (35)$$

If α , whose NDA value is $1/(16\pi^2)$, is not unnaturally small and $\Lambda_5 L > 1$, the contribution to the energy of the new operator is safely small. Notice that, contrary to the naive expectation, we did not loose too much in our perturbation parameter because of the fact that we needed to include the operator Eq. (20) to stabilize the soliton. If we would have found a solution for $\alpha = 0$, its size would have been given by $\rho_0 \sim L$ and we would have obtained, instead of Eq. (35), a perturbative expansion parameter of order $1/(\Lambda_5 L)$.

We can safely neglect any d -dimensional bulk and brane-localized operators with, respectively, $d \geq 6$ and $d \geq 5$. The only operator that could be important in our analysis is the five-dimensional Chern-Simons (CS) term which, by NDA, is suppressed by $M_5/\Lambda_5 \sim \alpha$, and therefore it can be as significant as Eq. (22). This term is however absent in our case due to its relation with the anomalies of the theory; it vanishes because $SU(2)$ is an anomaly-free

group. Even in models where the CS term can be present, like for example in $U(2)$ gauge theories, its effect could be as small as that of dimension 6-operators [4], even though the situation is not completely clear. We leave the analysis of the CS effects for the future.

The issue of calculability is particularly compelling in our case, and it is different from what happens in the 4D Skyrme model in which all operators give comparable contributions.⁴ The problem, in the case of the 4D skyrmions, comes from the fact that a higher-dimensional operator of the form $\alpha (\partial U)^4$ must be added to the Goldstone kinetic term $F_\pi^2 (\partial U)^2$ in order to obtain a regular solution of the EOM [1, 14]. The value of α estimated by NDA is F_π^2/Λ^2 , where $\Lambda \sim 4\pi F_\pi$ is the cut-off of the chiral lagrangian. We then see that F_π^2 factorizes in front of the action and the cut-off Λ is the only dimensionful quantity which enters into the EOM, setting the radius of the skyrmion to $\rho_0 \sim 1/\Lambda$. In our 5D case also a higher-dimensional operator is needed to stabilize the soliton. Its coefficient is given by $\alpha \sim M_5/\Lambda_5$ and then also the 5D coupling M_5 factorizes in front of the action. Nevertheless, our soliton configuration not only depends on the 5D cut-off Λ_5 , but also on the conformal length L . This is the crucial difference from the 4D Skyrme model; a combination of L and Λ_5 sets the size of the 5D soliton fixing it to a scale larger than $1/\Lambda_5$.

3 Conclusions

We have studied skyrmion configurations arising from compact five dimensional models. We have shown that the size of these skyrmions is stabilized by the presence of IR-boundary kinetic terms. This size is always larger than the inverse of the 5D cut-off scale $1/\Lambda_5$, and therefore consistent within our 5D effective theory. This is different from previous 5D models [3, 4] in which the skyrmion size was found to be of order $1/\Lambda_5$. We have numerically obtained the skyrmion configurations for different values of α (the coefficient of the IR-boundary kinetic term), and calculated their size and energy. Although the size of the skyrmions depends strongly on α , their energy is quite model-independent. By varying α these skyrmion configurations smoothly interpolates between small 4D instantons and 4D skyrmions made of Goldstones and a massive gauge boson.

The existence of stable skyrmion configurations in extra dimensional models rises different phenomenological issues. For example, theories of electroweak symmetry breaking arising from extra dimensions will have this type of stable configurations, and therefore the analysis of their cosmological consequences are of great importance. For 5D Higgsless [7] or composite

⁴When this is the case, not only the effective theory does not give us any quantitative information on the soliton, but also its very existence is doubtful.

Higgs models [15] in which $F_\pi = 246/\epsilon$ GeV (where $\epsilon \leq 1$) and $m_\rho \simeq 1.2/\epsilon$ TeV, we find a soliton energy $M_N \sim (2.5 - 3)/\epsilon$ TeV. These TeV stable particles are possible dark matter candidates, therefore the precise determination of their relic abundances is of important phenomenological interest. These configurations can also be useful in holographic models of 4D strong interactions. The AdS/CFT correspondence tells us that these 5D solitons correspond to the "baryons" of the dual theory. The 5D AdS model studied here has already been proposed as a holographic model of QCD, giving predictions for the meson spectrum and couplings in good agreement with the experimental data [8]. The skyrmion found here gives an approximate value for the mass of the proton. We find $M_N \simeq 500 - 650$ GeV, too low compared with the experimental value ~ 1 GeV. We must notice, however, that in our analysis we have not included the 5D CS term, responsible in holographic QCD models of the Wess-Zumino-Witten (WZW) term. It is known that the presence of the WZW term has important impact on the skyrmion mass [16], so we can expect that including the CS in our analysis can enhance our prediction of the proton mass. These and other phenomenological questions deserve a further analysis that we leave for a future work.

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Appendix: The instanton energy in compact and warped spaces

In the 2D notation used in this paper the BPST instanton with center at $(\mathbf{x} = 0, z = z_{\text{IR}})$ and size ρ corresponds to [17]

$$\begin{aligned} \phi_1 &= r \frac{\partial_z \Phi}{\Phi} \equiv \bar{\phi}_1(\rho), \quad \phi_2 = -1 - r \frac{\partial_r \Phi}{\Phi} \equiv \bar{\phi}_2(\rho), \\ A_1 &= \frac{\partial_z \Phi}{\Phi} \equiv \bar{A}_1(\rho), \quad A_2 = -\frac{\partial_r \Phi}{\Phi} \equiv \bar{A}_2(\rho), \end{aligned} \tag{36}$$

where

$$\Phi = \frac{1}{\rho^2 + r^2 + (z - z_{\text{IR}})^2}. \tag{37}$$

The singular instanton of size $\rho \rightarrow 0$ solves, up to a gauge transformation, the variational problem defined by the energy Eq. (11) and the boundary conditions Eqs. (13) and (15).

Indeed, it has the same topological charge ($Q = 1$) of the solution we were looking for and, since its 2D energy density is a δ -function at $r = 0$, $z = z_{\text{IR}}$, it has the minimal energy $E = 8\pi^2 M_5$ which saturates the lower bound of Eq. (17).

We can explicitly check that the instanton Eq. (36) fullfills, for any ρ , the boundary conditions at $r = 0$ and $z = z_{\text{IR}}$ given in Eqs. (13) and (15). At the other two boundaries the instanton of size $\rho \rightarrow 0$ has different boundary conditions:

$$z = z_{\text{UV}} : \begin{cases} \phi = \bar{\phi}(0) = -ie^{i\beta} \\ A_1 = \bar{A}_1(0) = \partial_1 \beta \\ \partial_{\bar{\mu}} A^{\bar{\mu}} = 0 \end{cases}, \quad r \rightarrow \infty : \begin{cases} \phi = \bar{\phi}(0) = -ie^{i\beta} \\ A_2 = \bar{A}_2(0) = \partial_2 \beta \\ \partial_{\bar{\mu}} A^{\bar{\mu}} = 0 \end{cases}, \quad (38)$$

where $\beta = 2 \arctan[r/(z_{\text{IR}} - z)]$. These boundary conditions are, as anticipated, topologically equivalent to those in Eq. (13) and then one can convert one into the other by a gauge transformation. Instead of gauge rotating the instanton of $\rho = 0$ to make it fulfill Eq. (13), it is more convenient to rephrase our original problem in the new gauge and the new boundary conditions Eq. (38).

Let us consider instantons of small but non-vanishing size. We would like to compute the energy $E(\rho)$ of such configurations. We already know that $E(\rho)$ will have an absolute minimum at $\rho = 0$; this means that a small instanton is subject to an "attractive force" which tends to shrink its size to zero. By computing $\partial_\rho E(\rho)$ one can measure the strength of this force. It is important to remark that there are two different effects which make the force arise. The first one, which is essentially local, is due to the curvature of the space and therefore it is only present in warped spaces. It pushes the instanton to be localized as much as possible at z_{IR} where the warp factor has its minimal value and then it is energetically favorable. The second effect is non-local and is due to the presence of the boundary at z_{UV} which confines the solution into a finite volume. At the more technical level the two effects come, respectively, from the fact that the instanton fails to fulfill the bulk EOM when the space is warped and the boundary condition Eq. (38) at z_{UV} . It cannot then be a stable configuration.

To compute $E(\rho)$ let us first try to substitute the instanton configuration in the energy functional Eq. (11). We call E_B the "bulk" contribution to the energy which we obtain in this way. We will see later that this is not the only contribution to $E(\rho)$. For a generic warp

factor we have ⁵

$$E_B(\rho) = 12 \pi^2 M_5 \int_{-L/\rho}^0 dy \frac{a(z_{\text{IR}} + \rho y)}{(1 + y^2)^{5/2}} = 12 \pi^2 M_5 \sum_n \rho^n c_n(\rho/L) \frac{a^{(n)}(z_{\text{IR}})}{n!}, \quad (39)$$

where we have expanded $a(z)$ in Taylor series around $z = z_{\text{IR}}$ and

$$c_n(\rho/L) = \int_{-L/\rho}^0 dy \frac{y^n}{(1 + y^2)^{5/2}}. \quad (40)$$

The leading term of E_B in the small ρ expansion comes from the $n = 0$ term which reads

$$12 \pi^2 M_5 c_0(\rho/L) \simeq 8 \pi^2 M_5 \left(1 - \frac{3 \rho^4}{8 L^4} + \mathcal{O}(\rho^6/L^6) \right), \quad (41)$$

where we have also expanded $c_0(\rho/L)$ around $\rho = 0$ and have kept, for future convenience, the first correction of order ρ^4 . A linear contribution to the energy can only come from the $n = 1$ term and is given by

$$- 4 \pi^2 M_5 \rho a^{(1)}(z_{\text{IR}}) (1 + \mathcal{O}(\rho^3/L^3)). \quad (42)$$

If $a^{(1)}$ is non zero, this term must be negative since the warp factor is a decreasing function of z . Therefore the above term gives, as expected, an attractive force which makes the instanton shrink. In the case of AdS, Eq. (25), we have $a^{(1)} = -1/L$ and we obtain the result of Eq. (19). For a space with $a^{(1)} = 0$ the linear contribution vanishes and the leading correction to the energy, which comes from the $n = 2$ term, is of $\mathcal{O}(\rho^2/L^2)$. Making the space less and less warped near z_{IR} we are increasing the power of ρ/L of the instanton energy, and therefore making the attractive force weaker and weaker at small ρ . This process however stops at order ρ^4/L^4 . The $n = 4$ term is proportional to $c_4 \sim \ln \rho$, and for $n > 4$ we have $c_n \sim 1/\rho^{n-4}$, so all terms $n \geq 4$ contribute at the order ρ^4/L^4 to the energy. Therefore we expect that the weakest possible force will be obtained from a energy of $\mathcal{O}(\rho^4/L^4)$ that is, as we will now show, what actually happens in the case of flat space.

For flat space only the $n = 0$ term appears in Eq. (39) and E_B is given by Eq. (41). We then immediately see that E_B cannot be the total instanton energy. If so, the point $\rho = 0$ would not be a minimum but a maximum, and we would not find an attractive but a repulsive force. The reason why there is an extra contribution to the energy is that the instanton of finite size does not respect the boundary condition at z_{UV} . Therefore, it does not belong to

⁵We are considering, everywhere in this Appendix, $z_{\text{UV}} \neq 0$ because for $z_{\text{UV}} = 0$ the energy of the instanton in the AdS slice diverges. Our numerical computations show however that the true solution is widely insensitive to the position of the UV-boundary and the results we derive here can be applied for $z_{\text{UV}} = 0$ as well.

the set of allowed field configurations on which our variational problem is defined. In order to find the instanton energy we need a different formulation of the variational problem in which the allowed field configurations do not necessarily respect the boundary condition. The latter will arise, like the EOM, from the minimization of a new energy functional which is now defined to act on a more general set of configurations, to which the instanton belongs. This new functional is given by Eq. (11) and the UV-boundary term ⁶

$$E_{\text{UV}} = 8\pi M_5 a(z_{\text{UV}}) \int_0^\infty dr \left[(\phi - \bar{\phi}(0))^* D_z \phi + (\phi - \bar{\phi}(0)) D_z \phi^* - \frac{r^2}{2} (A_1 - \bar{A}_1(0)) F_{12} \right]_{z_{\text{UV}}} . \quad (43)$$

It is easily understood why the addition of E_{UV} makes the UV boundary conditions arise as EOM. The variation of Eq. (11) and (43) gives, in addition to the usual bulk terms whose cancellation will give rise to the EOM, localized UV-terms of the form $(\phi - \bar{\phi})^* \delta(D_z \phi)$ and $(A_1 - \bar{A}_1) \delta(F_{12})$. Requiring those terms to vanish enforces the boundary conditions in Eq. (38).

We can finally compute the energy $E(\rho)$ of the finite size instanton. It is given by the sum of E_B in Eq. (39) and E_{UV} in Eq. (43), in which of course we have to substitute the instanton configuration. The latter term reads

$$E_{\text{UV}}(\rho) = 5\pi^2 M_5 a(z_{\text{UV}}) \frac{\rho^4}{L^4} , \quad (44)$$

and gives, for a generic warped metric, a ρ^4 contribution to the energy. This contribution does not affect the result in the case of the AdS slice, since it is subleading, but it is crucial in flat space as it corrects the negative sign which we found in Eq. (41). Adding E_B and E_{UV} we obtain Eq. (18).

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⁶We can obtain this result by using Lagrange Multipliers that allows to include boundary constraints in the functional to minimize.

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